The batch-type neural test model:
A latent rank model with the mechanism of generative topographic mapping

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1. Introduction

Item reference profile (IRP) that represents the transition of the correct answer ratio through the latent rank scale for each item is obtained in the NTT analysis. However, the IRP estimate differs in each calculation. Differences between estimates can be obtained even when the estimation process is rerun under the same settings. The purpose of this study is to investigate a batch-type version of the NTT model to obtain invariant IRP estimates. This model uses the mechanism of the generative topographic mapping (GTM; Bishop, Svensen, & Williams, 1998). In this paper, we propose an estimation framework for obtainining smooth and invariant IRPs by retrofitting the EM algorithm with an elastic mechanism.

2. Method

2.1 Statistical Learning Framework

Let us assume that the sample size is N, the number of items is n, and the response data of the N examinees for the n items is $\boldsymbol{U} = \{u_{ij}\}$ $(N \times n)$. Let us also assume that $\boldsymbol{Z} = \{z_{ij}\}$ $(N \times n)$ is the missing indicator matrix. Also assume that the number of latent ranks is Q and the q-th latent rank is denoted R_q $(q = 1, \cdots, Q)$. In addition, $\boldsymbol{F} = \{f_{iq}\}$ $(N \times Q)$ is the rank membership indicator (RMI) matrix. Let us further assume that $\boldsymbol{V} = \{v_{qj}\}$ $(Q \times n)$ is the rank reference matrix (RRM). The j-th row vector of the RRM, $\boldsymbol{v}_j = \{v_{jq}\}$ $(Q \times 1)$, is the item reference profile (IRP) of item j. The estimation framework for the batch-type NTT model is given by

$$\text{Obtain } \boldsymbol{Z} \text{ from } \boldsymbol{U}. \tag{1}$$

$$\text{Define } \boldsymbol{V}^{(0)}. \tag{2}$$

$$\text{For } (t{=}1;\ t \leq T;\ t = t+1) \tag{3}$$

$$\text{— Obtain } \boldsymbol{F}^{(t)} \text{ by using } \boldsymbol{U} \text{ and } \boldsymbol{V}^{(t-1)}. \tag{4}$$

$$\text{— Obtain } \boldsymbol{E}^{(t)} \text{ by using } \boldsymbol{F}^{(t)}. \tag{5}$$

$$\text{— Obtain } \boldsymbol{V}^{(t)} \text{ by using } \boldsymbol{E}^{(t)}. \tag{6}$$

2.2 Expected Log-Likelihood

The likelihood that $\boldsymbol{U}$ is observed provided that the RRM $\boldsymbol{V}$ and RMI $\boldsymbol{F}$ are given is

$$\ln p(\boldsymbol{V}|\boldsymbol{U}) = \sum_{i=1}^{N}\sum_{q=1}^{Q}\sum_{j=1}^{n} f_{iq}^{(t)} z_{ij}\big\{u_{ij}\ln v_{jq} + (1-u_{ij})\ln(1-v_{jq})\big\} + \ln p(\boldsymbol{V}) + const, \tag{7}$$

where $f_{iq}^{(t)}$ is the posterior probability that examinee i belongs to latent rank R_q at the t-th EM cycle. That is,

$$p(f_{iq}|\boldsymbol{u}_i, \boldsymbol{V}^{(t-1)}) = f_{iq}^{(t)} = \frac{p(\boldsymbol{u}_i|\boldsymbol{v}_q^{(t-1)})\pi_{iq}}{\sum_{q'=1}^{Q} p(\boldsymbol{u}_i|\boldsymbol{v}_{q'}^{(t-1)})\pi_{iq'}} \quad (i = 1, \cdots, N;\ q = 1, \cdots, Q), \tag{8}$$

where π_{iq} is the prior probability of f_{iq}.

2.3 Scale Elasticity and Weakly Ordinal Alignment Condition

Shojima (2008b) defined weakly and strongly ordinal alignment conditions for expressing the degree of ordinality of the latent scale. The weak condition is satisfied when the test reference profile (TRP) is monotonically increasing, while the strong condition requires all the IRPs to be monotonic.

It is necessary to improve the elasticity of the scale to satisfy the weak condition in the NTT model with the GTM mechanism. Therefore, it is of great significance to obtain the elastic rank membership indicator $\boldsymbol{E}^{(t)}$ from the rank membership indicator (RMI) $\boldsymbol{F}^{(t)}$. One useful way to obtain the elastic RMI is $\boldsymbol{E}^{(t)} = \boldsymbol{F}^{(t)}\boldsymbol{G}$. That is, the elastic RMI is given by weighting the RMI with a smoothing matrix $\boldsymbol{G}$. As for the linear filter $\boldsymbol{\phi}_b$ in $\boldsymbol{G}$ being desirable for the batch-type NTT model are

$$\boldsymbol{\phi}_3 = \Big\{\frac{1-\phi}{2}, \phi, \frac{1-\phi}{2}\Big\} \quad \Big(0 \le \frac{1-\phi}{2} \le \phi \le 1\Big), \tag{9}$$

and

$$\boldsymbol{\phi}_5 = \Big\{\frac{1-\phi_1-2\phi_2}{2}, \phi_2, \phi_1, \phi_2, \frac{1-\phi_1-2\phi_2}{2}\Big\} \quad \Big(0 \le \frac{1-\phi_1-2\phi_2}{2} \le \phi_2 \le \phi_1 \le 1\Big). \tag{10}$$

Furthermore, it is desirable to use a different linear filter according to the number of latent ranks. The parameter ϕ in Equation (9) that we recommend is

$$\phi = \begin{cases} 1.05 - 0.05Q & (1 \le Q \le 5) \\ 1.00 - 0.04Q & (5 \le Q \le 10) \\ 0.80 - 0.02Q & (10 \le Q \le 20) \end{cases}. \tag{11}$$

Using the elastic RMI weighted by the smoothing matrix, we can reconstruct the expected log-likelihood obtained in the t-th E-step (Equation 7) as follows:

$$\ln p(\boldsymbol{V}|\boldsymbol{U}) = \sum_{i=1}^{N}\sum_{q=1}^{Q}\sum_{j=1}^{n} e_{iq}^{(t)} z_{ij}\big\{u_{ij}\ln v_{jq} + (1-u_{ij})\ln(1-v_{jq})\big\} + \ln p(\boldsymbol{V}) + const. \tag{12}$$

Then, in the M-steps, the expected log-likelihood is optimized with respect to the structural parameters. Equation (12) can be maximized with respect to each RRE when the prior probability of each RRE is a constant. The RRE estimate at the t-th EM cycle is obtained as

$$v_{jq}^{(t)} = \frac{\sum_{i=1}^{N} z_{ij}u_{ij}e_{iq}^{(t)}}{\sum_{i=1}^{N} z_{ij}e_{iq}^{(t)}} = \frac{\sum_{i=1}^{N} z_{ij}u_{ij}\boldsymbol{g}'_q\boldsymbol{f}_i^{(t)}}{\sum_{i=1}^{N} z_{ij}\boldsymbol{g}'_q\boldsymbol{f}_i^{(t)}}. \tag{13}$$

2.4 Effective Degrees of Freedom and Minimum Information Estimation of Smoothing Matrix

The effective degree of freedom (Hastie, Tibshirani, & Friedman, 2001) of item j of the batch-type NTT model is given by

$$edf_j = n - \mathrm{tr}\ \boldsymbol{G}. \tag{14}$$

The χ^2 statistic of the batch-type NTT model can then be computed as

$$C_j = 2\{\ln p(\hat{\boldsymbol{v}}_{Bj}|\boldsymbol{u}_j) - \ln p(\boldsymbol{v}_j^{(t)}|\boldsymbol{u}_j)\}, \tag{15}$$

where $\hat{\boldsymbol{v}}_{Bj}$ $(n \times 1)$ is the IRP estimate of the benchmark model for item j. In addition, some information criteria for each item can be evaluated from the above statistic.

The minimum information estimation (Akaike, 1974) is a method of estimating the model parameters by minimizing an information criterion. In the case of the batch-type NTT model, it is possible to explore an optimal smoothing matrix by setting an information criterion as the objective function. Let the following step be inserted after Line (4):

$$\text{— Obtain } \boldsymbol{G}^{(t)} \text{ by using } \boldsymbol{F}^{(t)} \text{ and } \boldsymbol{V}^{(t-1)}. \tag{16}$$

Given the RRM estimate at the $t-1$-th period ($\boldsymbol{V}^{(t-1)}$) and the RMI estimate at the t-th period ($\boldsymbol{F}^{(t)}$), the information criterion can be written as

$$IC(\boldsymbol{G}|\boldsymbol{V}^{(t-1)}, \boldsymbol{F}^{(t)}) = -2\sum_{i=1}^{N}\sum_{q=1}^{Q}\sum_{j=1}^{n} \boldsymbol{g}'_q\boldsymbol{f}_i^{(t)} z_{ij}\big\{u_{ij}\ln v_{jq}^{(t-1)} + (1-u_{ij})\ln(1-v_{jq}^{(t-1)})\big\} + kn\mathrm{tr}\ \boldsymbol{G}. \tag{17}$$

The smoothing matrix that minimizes Equation (17) under the linear constraints is the estimate of the smoothing matrix at the t-th period, $\boldsymbol{G}^{(t)}$. Consequently, the RRM estimate at the t-th period is then obtained by

$$v_{jq}^{(t)} = \frac{\sum_{i=1}^{N} z_{ij} u_{ij} e_{iq}^{(t)}}{\sum_{i=1}^{N} z_{ij} e_{iq}^{(t)}} = \frac{\sum_{i=1}^{N} z_{ij} u_{ij} \boldsymbol{g}_q^{(t)\prime} \boldsymbol{f}_i^{(t)}}{\sum_{i=1}^{N} z_{ij} \boldsymbol{g}_q^{(t)\prime} \boldsymbol{f}_i^{(t)}}. \tag{18}$$

2.5 Stopping Rule and Strongly Ordinal Alignment Condition

When the IRPs of all the items are monotonic, the TRP is consequently monotonic. In this case, the strongly ordinal alignment condition is satisfied. It is possible to satisfy the strongly ordinal alignment conditon by imposing the monotonic increase constraint on the IRPs of all the items.

3. Analysis

Figure 1 shows the IRPs of the first 10 items. Table 1 lists the model-fit indices for the whole test. The rank membership profiles (RMPs; Shojima, 2008b) of examinees 1–10 out of the 5000 samples are shown in Figure 2. Some additional information obtained by the NTT analysis is shown in Figure 3.

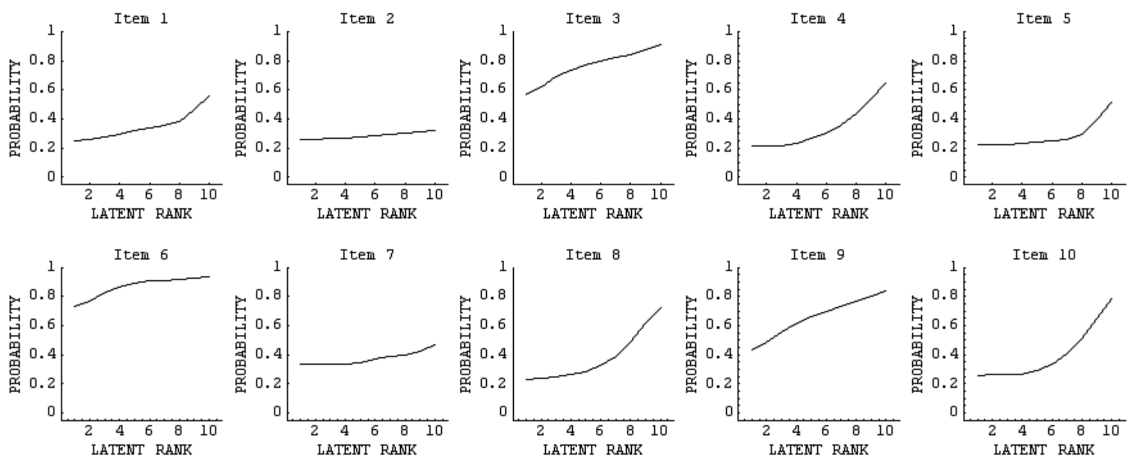


Figure 1: Item Reference Profiles of Items 1–10 ($Q = 10$, Fixed Smoothing Matrix)

Table 1: Test Fit Indices ($Q = 10$, Fixed Smoothing Matrix)

Index	Value
$\chi^2_{1004.5}$	1841.95
NFI	0.897
RFI	0.878
IFI	0.964
TLI	0.957
CFI	0.964
RMSEA	0.010
AIC	−527.05
CAIC	−8078.07
BIC	−7073.57

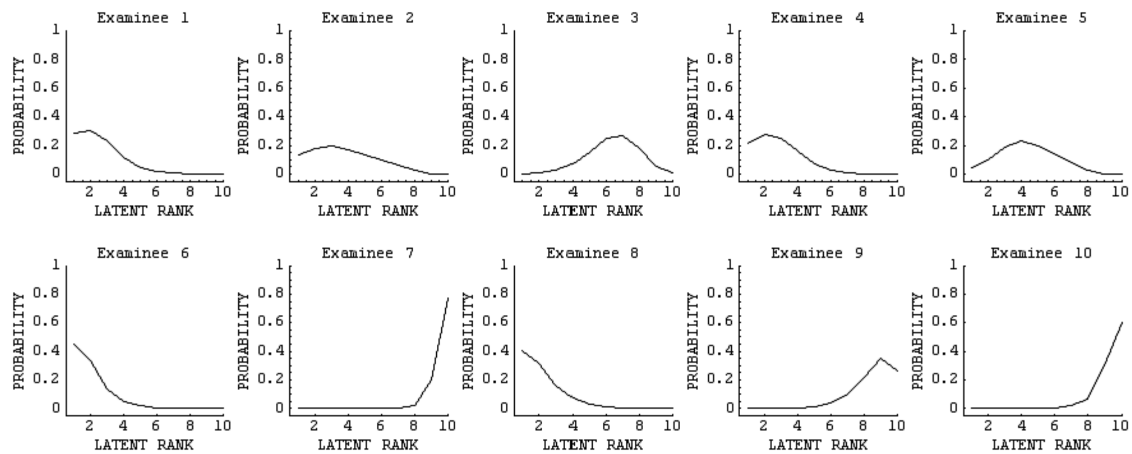
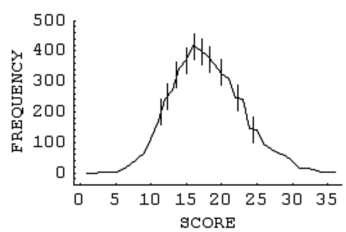


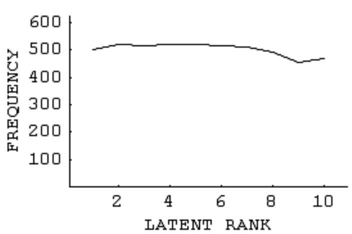
Figure 2: Rank Membership Profiles of Examinees 1–10 ($Q = 10$, Fixed Smoothing Matrix)

(a) Test Reference Profile (TRP)

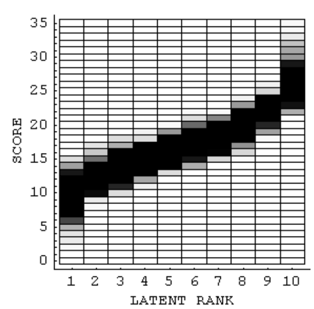


(b) TRP on Score Distribution

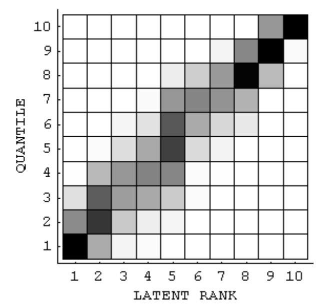
(c) Latent Rank Distribution



(d) Rank Membership Distribution



(e) Rank-Score Scatter Plot



(f) Rank-Quantile Scatter Plot

Figure 3: TRP, LRD, RMD, and Scatter Plots ($Q = 10$, Fixed Smoothing Matrix)

References

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Shojima, K. (2008b) Neural test theory: A latent rank theory for analyzing test data. *DNC Research Note*, 08-01.